Vector Addition and Subtraction

Key Ideas

- Graphically, vectors are summed in a "head to tail" manner.
- Vectors exist independently of coordinate systems. Vector addition does not require a coordinate system. However, in solving physics problems, we will be using a convenient coordinate system.
- Like the algebra of real numbers, the algebra of vectors exhibits many of the same mathematical properties.

Learning Objectives

By completing this section, you should:

- be able to graphically represent a vector sum by placing vectors head to tail.
- be able to graphically represent a vector sum using the parallelogram method.
- be able to evaluate a vector sum using only geometry and trigonometry, including the law of cosines and the law of sines.
- understand and apply the properties of vector algebra.
- be able to perform vector addition using scalar components of Cartesian unit-vectors or as a vector magnitude with an angle with respect to a known axis.

Previously, we added orthogonal vector components that are the legs of a right triangle in order to obtain a vector that is the hypotenuse of that same triangle. When working with unit vectors, we have multiplied a vector by a scalar to change its length, and we noted that multiplication by a negative scalar reverses its direction. Now we will consider vector addition more broadly. First we will present an overview of vector addition, and then we will focus on operating in the context of a coordinate system.

Vector Addition Basics

Graphical Addition of Vectors

To improve your understanding of vectors, it is helpful to add vectors using the **head to tail method**, as shown in the figure below. The vector $\vec{b}$ in the image on the left is directed at angle θ measured counterclockwise from vector $\vec{a}$. This angle is defined with the vectors "tail to tail." The sum of two vectors is the **resultant vector** $\vec{c}$.

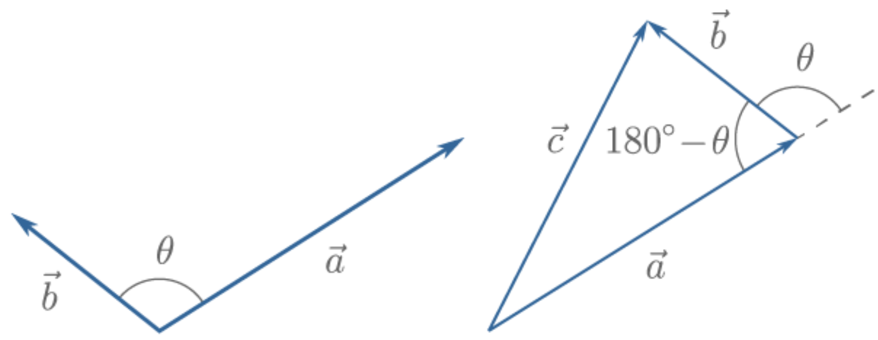


Figure 2.10 The angle between two vectors is defined with the vectors placed tail to tail as shown on the left. When adding vectors, they must be placed head to tail, as shown on the right, where the tail of $\vec{b}$ is placed at the head of $\vec{a}$. The resultant vector, $\vec{c}$ begins at the tail of $\vec{a}$ and ends at the head of $\vec{b}$.

The **parallelogram method** is another way of viewing the construction of the resultant sum of two vectors. Starting from a particular point, the two vectors are placed head to tail. Starting from that same point, the vectors are placed head to tail once more, but in the reverse order. The result is a parallelogram, and the resultant is the diagonal of the parallelogram with its tail placed at the original starting point. See the image below.

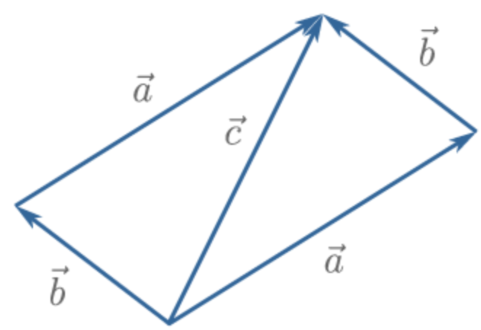


Figure 2.11 Starting from a common point, the vector sums $\vec{a} + \vec{b}$ and $\vec{b} + \vec{a}$ form the sides of a parallelogram. The resultant $\vec{c}$ is the diagonal where its tail is placed at the common starting point.

Fundamental Properties of Vectors

Many of the fundamental properties that you are familiar with in working with numbers are also true for vectors.

Zero Vector

There is a **zero vector** that may be added to a vector that does not change that vector. This is known as the **additive identity**, because

$$\vec{a} + \vec{0} = \vec{a} \tag{2.10}$$

The zero vector is different from other vectors because its direction is undefined. All of its vector components are zero, and its magnitude is zero. It cannot be represented directly with a graphic representation. It appears naturally in vector expressions such as

$$\vec{a} - \vec{a} = \vec{0} \tag{2.11}$$

Commutative and Transitive Properties

As we saw in Figure 2.11, for any two vectors $\vec{a}$ and $\vec{b}$, the **commutative** property indicates that the order when summing does not matter:

$$\vec{a} + \vec{b} = \vec{b} + \vec{a} \tag{2.12}$$

The **transitive** property states that,

$$\text{if } \vec{u} = \vec{v} \text{ and } \vec{v} = \vec{w}\,, \text{ then } \vec{u} = \vec{w} \tag{2.13}$$

which was applied in Figure 2.11 above as,

$$\begin{aligned} \text{If } \vec{a} + \vec{b} = \vec{b} + \vec{a} \quad &\text{and} \quad \vec{a} + \vec{b} = \vec{c}, \\ \text{then,} \quad &\vec{b} + \vec{a} = \vec{c} \end{aligned}$$

where $\vec{c}$ is the resultant of the sum.

With the parallelogram method of vector addition, the vector sum is conducted with both orders of addition, and the resultant is a diagonal. Many students find this to be a very useful visualization.

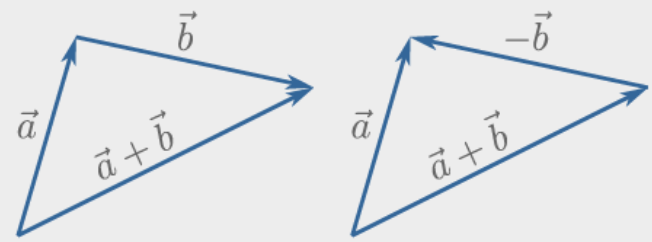
Associative Property

For any three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$, the **associative** property indicates that the order in which the addition operations are performed in a sum of three vectors does not matter:

$$\left(\vec{a} + \vec{b}\right) + \vec{c} = \vec{a} + \left(\vec{b} + \vec{c}\right) \tag{2.14}$$

Additive Inverse

The **additive inverse** of vector $\vec{b}$ is $-\vec{b}$, and it is represented by a vector where the head and the tail have been interchanged. As shown in the image below, the addition of $-\vec{b}$ reverses the effect of adding $\vec{b}$.



The image on the left shows the sum of $\vec{a}$ and $\vec{b}$ to produce the resultant vector $\vec{a}+\vec{b}$. The image on the right shows the sum

$$\left(\vec{a}+\vec{b}\right)+\left(-\vec{b}\right)=\vec{a}$$

The method you will most often use for adding vectors is adding the vector components. These components are actually projections of the vector onto the coordinate system axes. These components completely describe any vector.

In the image below, two vectors and their resultant sum are decomposed into their Cartesian components.

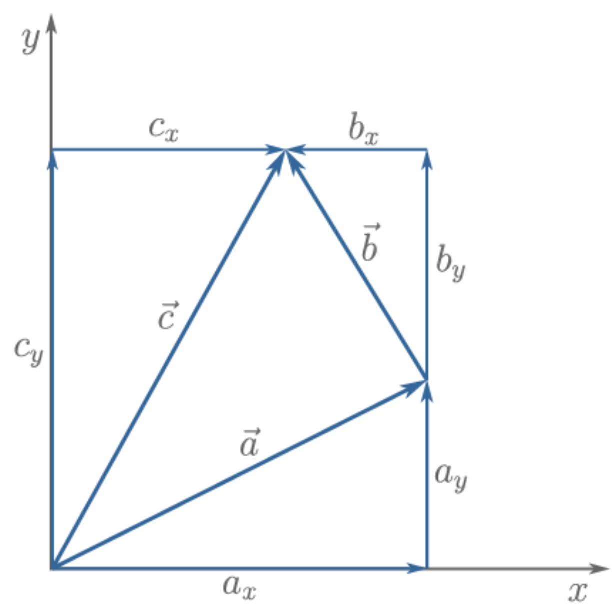


Figure 2.12 The vector sum $\vec{c}=\vec{a}+\vec{b}$ is shown with all three vectors decomposed into their Cartesian components